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and Inelastic Strain**

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Reciprocal Theorem of Residual Stress and Inelastic Strain

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Introduction:

We consider a body to have an initial stress field τ_{ij}^I . When the body is loaded beyond the elastic range, plastic strain e_{ij}^P occurs in some part of the body. If the body is then unloaded, the plastic strain remains and causes a residual stress field τ_{ij}^R . Now consider the load applied again. This applied load causes a stress τ_{ij}^A . There is no increase of plastic strain during this process of unloading and reloading. Now the stress field is the sum of the initial, the residual and the applied stress

$$\tau_{ij} = \tau_{ij}^I + \tau_{ij}^R + \tau_{ij}^A \quad (1)$$

As shown by Lin, 1968, using this analogy between plastic strain and applied force, we can find τ_{ij}^R in terms of e_{ij}^P in the body. As the load is increased, τ_{ij}^A increases. Then from the incremental stress strain relation of the material, the incremental plastic strains in different parts of the body are found (Lin, 1968). In this inelastic analysis, the relation between the plastic strain and the residual stress is needed. Plastic strain is one type of inelastic strain. Hence the reciprocal relation of inelastic strain and residual stress is very useful in numerical inelastic analyses.

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Derivation of the Reciprocal Relation for Inelastic Bodies:

Referring to a set of rectangular coordinates (x_1, x_2, x_3) , the equilibrium condition is given as

$$\tau_{ij,j} + F_i = 0 \quad (2)$$

in the interior of the volume V , and

$$S_i = \tau_{ij} v_j \quad (3)$$

on the boundary Γ where τ_{ij} is the stress component, F_i the body force per unit volume along x_i -direction, S_i is the traction per unit area along x_i -axis and v_j is the direction cosine of the exterior normal of the boundary with x_j -axis. The repetition of the subscript denotes summation from one to three, and the subscript after comma denotes differentiation with respect to that axis. Multiplying (2) and (3) by a virtual displacement u_i^* and integrating yields

$$\begin{aligned} \int F_i u_i^* dv + \int S_i u_i^* d\Gamma &= - \int \tau_{ij,j} u_i^* dv + \int \tau_{ij} v_j u_i^* d\Gamma \\ &= \int \tau_{ij} u_{i,j}^* dv = \int \tau_{ij} e_{ij}^* dv \end{aligned} \quad (4)$$

The stress-strain relation of an elastic body may be written as

$$\tau_{ij} = c_{ijkl} e_{kl} \quad (5)$$

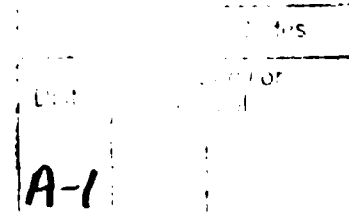
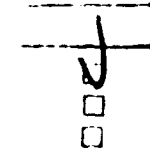
where c_{ijkl} is the elastic constant.

$$\int \tau_{ij} e_{ij}^* dv = \int c_{ijkl} e_{kl} e_{ij}^* dv = \int \tau_{kl}^* e_{kl} dv \quad (6)$$

From (4) and (6)

$$\int F_i u_i^* dv + \int S_i u_i^* d\Gamma = \int F_i^* u_i d\Gamma + \int S_i^* u_i d\Gamma \quad (7)$$

This is Betty's reciprocal theorem for elastic bodies.



Now consider the presence of inelastic strain e_{ij}'' which may be a combination of plastic strain, creep strain and thermal strain. The total strain e_{ij} is composed of the elastic strain e_{ij}' and inelastic strain e_{ij}''

$$e_{ij}' = e_{ij} - e_{ij}''$$

$$\tau_{ij} = c_{ijkl} (e_{kl} - e_{kl}'') \quad (8)$$

Substituting this into (2) and (3) yields

$$c_{ijkl} (e_{kl,j} - e_{kl,j}'') + F_i = 0 \quad (9)$$

$$S_i = c_{ijkl} (e_{kl} - e_{kl}'') v_j \quad (10)$$

It is seen that $c_{ijkl} e_{kl,j}''$ and $c_{ijkl} e_{kl}'' v_j$ have the same effect on the strain distribution c_{ij} as F_i and S_i respectively and hence they are denoted respectively as \underline{F}_i and \underline{S}_i (Lin, 1968).

Now consider an inelastic strain $e_{ij}''^I$ in region I giving \underline{F}_i^I and \underline{S}_i^I . The displacement, strain and stress in another region II caused by $e_{ij}''^I$ are denoted by $u_i^{II,I}$, $e_{ij}^{II,I}$ and $\tau_{ij}^{II,I}$ respectively as shown in Fig. 1.

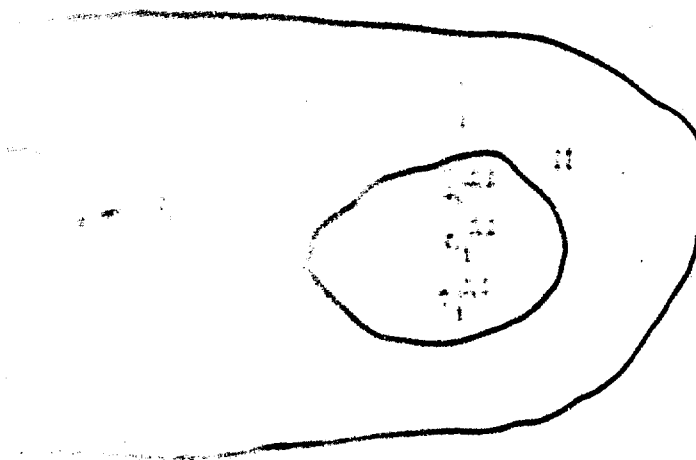


Diagram showing the nucleus and cytoplasm in II

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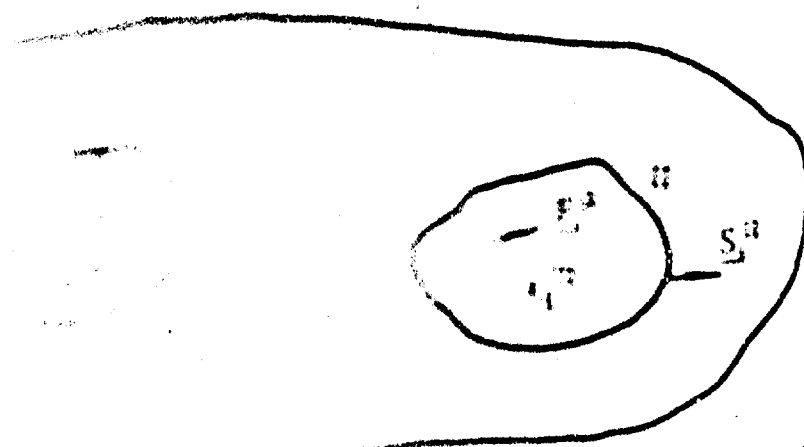


Diagram showing the nucleus and cytoplasm in II

Diagram showing the nucleus and cytoplasm in II

Applying Betty's Reciprocal Theorem (7) to Figs. 1 and 2 gives

$$\begin{aligned} \int_{V^I} \underline{F}_i^I u_i^{I,II} dv + \int_{\Gamma^I} \underline{S}_i^I u_i^{I,II} d\Gamma \\ = \int_{V^{II}} \underline{F}_i^{II} u_i^{II,I} dv + \int_{\Gamma^{II}} \underline{S}_i^{II} u_i^{II,I} d\Gamma \end{aligned}$$

Replacing \underline{F}_i by $-c_{ijkl} e_{kl,j}''$ and \underline{S}_i by $c_{ijkl} e_{kl}'' v_j$ yields

$$\begin{aligned} -\int_{V^I} c_{ijkl} e_{kl,j}'' u_i^{I,II} dv + \int_{\Gamma^I} c_{ijkl} e_{kl}'' v_j u_i^{I,II} d\Gamma \\ = -\int_{V^{II}} c_{ijkl} e_{kl,j}'' u_i^{II,I} dv + \int_{\Gamma^{II}} c_{ijkl} e_{kl}'' v_j u_i^{II,I} d\Gamma \end{aligned} \quad (11)$$

Using divergence theorem and noting $c_{ijkl} = c_{ijkl}$ Eq (11) becomes

$$\begin{aligned} \int_{V^I} c_{ijkl} e_{kl}'' e_{ij}^{I,II} dv = \int_{V^{II}} c_{ijkl} e_{kl}'' e_{ij}^{II,I} dv \\ \int_{V^I} \tau_{kl}^{I,II} e_{kl}'' dv = \int_{V^{II}} \tau_{kl}^{II,I} e_{kl}'' dv \end{aligned} \quad (12)$$

For the case $v^I = v^{II}$ and $e_{kl}'' \neq e_{kl}''$, the relation (12) still holds.

$\tau_{ij}^{I,II}$ is the stress in region I caused by inelastic strain e_{ij}'' in region II and $\tau_{ij}^{II,I}$ is the stress in region II caused by inelastic strain e_{ij}'' in region I. If regions I and II have no overlap, the stress $\tau_{kl}^{I,II}$ and $\tau_{kl}^{II,I}$ are residual stresses. If there is overlap as shown in Fig. 3. The residual stress in the overlapped region, denoted by $v^I \cap v^{II}$, is then

$$\begin{aligned} \tau_{kl,r}^{I,II} &= \tau_{kl}^{I,II} - c_{klmn} e_{mn}'' \\ \tau_{kl,r}^{II,I} &= \tau_{kl}^{II,I} - c_{klmn} e_{mn}'' \end{aligned} \quad (13)$$

This gives

$$\int_{V^I} \tau_{kl,r}^{I,II} e_{kl}'' dv = \int_{V^I - v^I \cap v^{II}} \tau_{kl}^{I,II} e_{kl}'' dv + \int_{v^I \cap v^{II}} \tau_{kl}^{I,II} e_{kl}'' dv$$

$$- \int_{v^I \cap v^{II}} c_{ijkl} e_{kl}^{II} e_{ij}^{II} dv \quad (14)$$

$$\begin{aligned} \int_{v^{II}} \tau_{kl,r}^{II,I} e_{kl}^{II} dv &= \int_{v^{II} - v^I \cap v^{II}} \tau_{kl,r}^{II,I} e_{kl}^{II} dv + \int_{v^I \cap v^{II}} \tau_{kl,r}^{II,I} e_{kl}^{II} dv \\ &- \int_{v^I \cap v^{II}} c_{ijkl} e_{kl}^{II} e_{ij}^{II} dv \end{aligned} \quad (15)$$

From (12) to (15)

$$\int_{v^I} \tau_{kl,r}^{I,II} e_{kl}^{II} dv = \int_{v^{II}} \tau_{kl,r}^{II,I} e_{kl}^{II} dv \quad (16)$$

This gives the reciprocal relation for inelastic strain and residual stress in solids.

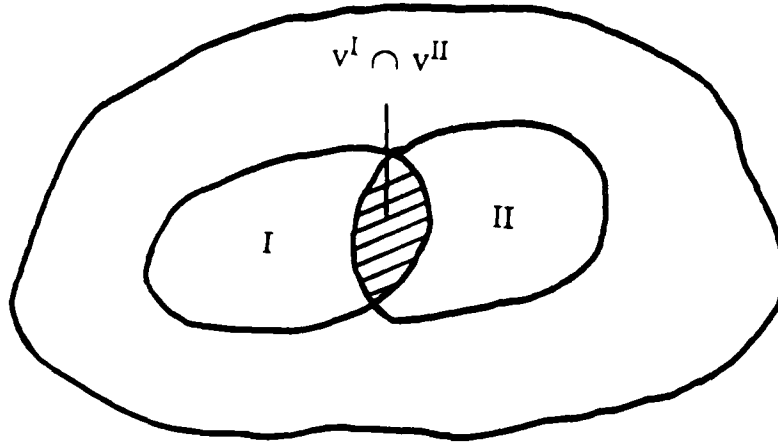


Fig. 3. v^I and v^{II} have an overlap $v^I \cap v^2$

Application to Fatigue Band Calculations:

A face-centered-cubic crystal has four slip planes, on each of which there are three slip directions giving twelve slip systems. Slip in a slip system depends on the resolved shear stress and is independent of the normal stress on the sliding plane (Taylor, 1938). Let us consider a most favorably oriented crystal in a F.C.C. polycrystal. When this polycrystal is loaded, slip may occur in some region of this crystal. After unloading, this slip remains and causes a residual stress. Imagine that the polycrystal is reloaded. This resolved shear stress in this crystal can be expressed as the sum of the initial resolved shear stress τ^I , the resolved shear stress due to the applied loading τ^A and the residual shear stress τ^R . If this resolved shear stress reaches the critical shear stress τ^c , i.e.

$$\tau = \tau^A + \tau^R = \tau^c \quad (17)$$

in some region, this region will start or continue to slide.

We consider three thin slices in a most favorably oriented crystal at the free surface of the polycrystal as shown in Fig.4 (Lin, 1974). An initial tensile strain $e_{\alpha\alpha}^I$ is assumed to exist in the slice R . This $e_{\alpha\alpha}^I$ causes an initial compressive stress in R , a positive resolved shear stress $\tau_{\alpha\beta}^I$ in P and a negative $\tau_{\alpha\beta}^I$ in Q (Lin and Lin, 1988). A cyclic loading of tension and compression on the polycrystal causes alternate sliding in P and Q causing the growth of an extrusion or an intrusion (Lin, 1977, Lin and Ito, 1969). An extrusion causes a tensile strain and hence a tensile stress along extrusion direction. This can cause the resolved shear stress in a second slip system to slide in R . This secondary slip system can slide concurrently with this primary slip system in P and Q . Let $\alpha\beta$ denote the primary slip system and $\xi\eta$, the second slip system. The plastic strains caused by slip in these two systems are denoted by $e_{\alpha\beta}^P$ and $e_{\xi\eta}^P$ respectively. The resolved shear stress $\tau_{\alpha\beta}^R(x)$ in $\alpha\beta$ -system at point x caused by a unit $e_{\alpha\beta}^P$ at \bar{x} in a differential volume dv is expressed as $G(x, \alpha\beta; \bar{x}, \alpha\beta)e_{\alpha\beta}^P dv$. If $e_{\xi\eta}^P$ also occurs, we can write

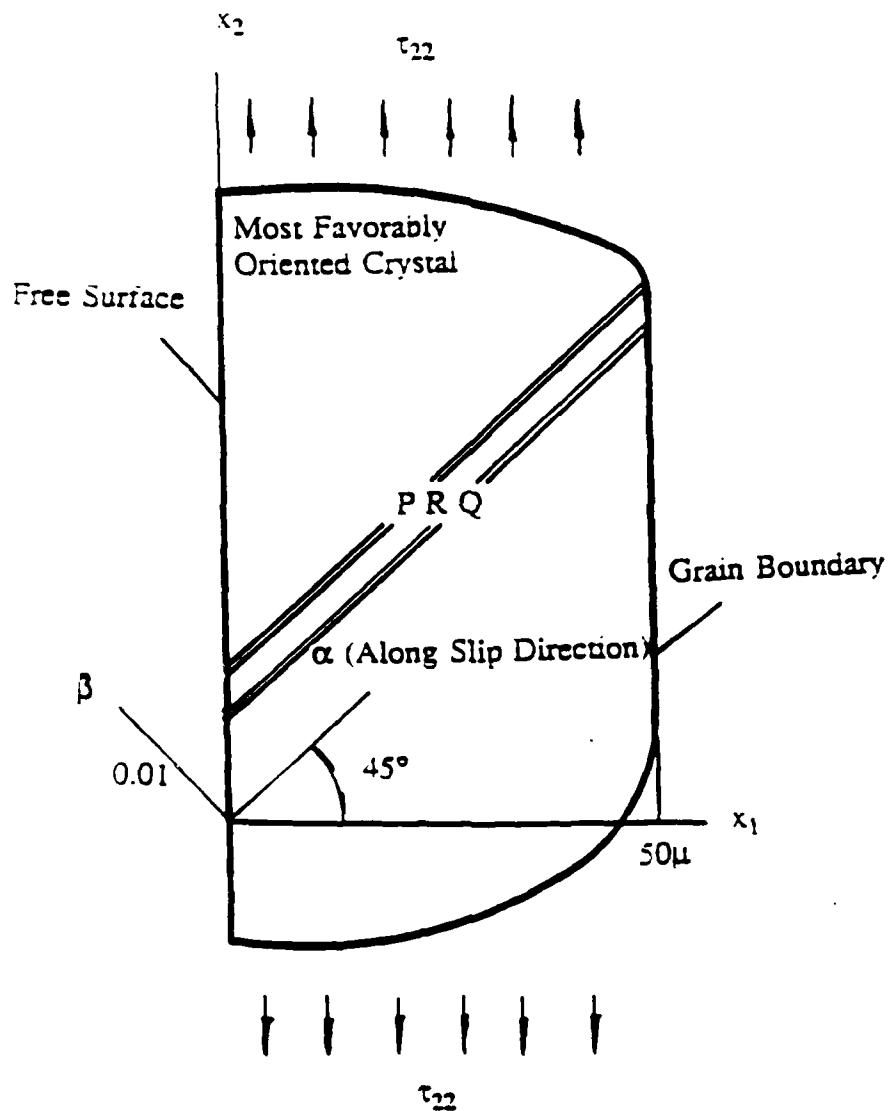


Fig.4 Most Favorable Oriented
Crystal at Free Surface

$$\tau_{\alpha\beta}^R(\mathbf{x}) = \int G(\mathbf{x}, \alpha\beta; \bar{\mathbf{x}}, \alpha\beta) e_{\alpha\beta}^R(\bar{\mathbf{x}}) dv + \int G(\mathbf{x}, \alpha\beta; \bar{\mathbf{x}}, \xi\eta) e_{\xi\eta}^P(\bar{\mathbf{x}}) dv \quad (19)$$

The two integrals are integrated over the regions with plastic strain. Similarly we can write

$$\tau_{\xi\eta}^R(\mathbf{x}) = \int G(\mathbf{x}, \xi\eta; \bar{\mathbf{x}}, \alpha\beta) e_{\alpha\beta}^P(\bar{\mathbf{x}}) dv + \int G(\mathbf{x}, \xi\eta; \bar{\mathbf{x}}, \xi\eta) e_{\xi\eta}^P(\bar{\mathbf{x}}) dv \quad (20)$$

In numerical calculation, the thin slices P, Q, R are divided into grids. In each grid, the $e_{\alpha\beta}^P$ or $e_{\xi\eta}^P$ is generally assumed to be constant or linear. The residual resolved shear stress due to plastic strains in the n th grid is written as

$$\tau_{\alpha\beta}^R(\mathbf{x}) = G(\mathbf{x}, \alpha\beta; n, \alpha\beta) e_{\alpha\beta}^P v_n + G(\mathbf{x}, \alpha\beta; n, \xi\eta) e_{\xi\eta}^P v_n \quad (21)$$

where

$$G(\mathbf{x}, \alpha\beta; n, \alpha\beta) = \frac{\int_{v_n} G(\mathbf{x}, \alpha\beta; \bar{\mathbf{x}}, \alpha\beta) e_{\alpha\beta}^P dv}{\int_{v_n} e_{\alpha\beta}^P(\bar{\mathbf{x}}) dv} \quad (22)$$

and $e_{\alpha\beta}^P$ is the average $e_{\alpha\beta}^P$ in the n th grid. Now we consider the residual stress in the m th grid corresponding to an assumed plastic strain distribution in the grid. As mentioned before, this distribution is generally assumed to be constant or linear. Multiplying (21) by $e_{\alpha\beta}^P dv$ in this grid and integrate

$$\begin{aligned} \int_{v_m} \tau_{\alpha\beta}^R(\mathbf{x}) e_{\alpha\beta}^P(\mathbf{x}) dv &= e_{\alpha\beta}^P v_n \int_{v_m} G(\mathbf{x}, \alpha\beta; n, \alpha\beta) e_{\alpha\beta}^P(\mathbf{x}) dv \\ &+ e_{\xi\eta}^P v_n \int_{v_m} G(\mathbf{x}, \alpha\beta; n, \alpha\beta) e_{\alpha\beta}^P(\mathbf{x}) dv \\ &= e_{\alpha\beta}^P v_n G(m, \alpha\beta; n, \alpha\beta) e_{\alpha\beta}^P v_n + e_{\xi\eta}^P v_m G(m, \alpha\beta; n, \xi\eta) e_{\alpha\beta}^P v_m \end{aligned} \quad (23)$$

where

$$G(m, \alpha\beta; n, \alpha\beta) = \frac{\int_{v_n} G(x, \alpha\beta; n, \alpha\beta) e_{\alpha\beta}^P(x) dv}{\int_{v_m} e_{\alpha\beta}^P(x) dv} \quad (24)$$

Writing the residual resolved shear stress in the m th grid as the left hand side of (23) divided by $e_{\alpha\beta}^P v_m = \int_{v_m} e_{\alpha\beta}^P(x) dv$, we have

$$\tau_{\alpha\beta}^R = G(m, \alpha\beta; n, \alpha\beta) e_{\alpha\beta}^P v_n + G(m, \alpha\beta; n, \xi\eta) e_{\xi\eta}^P v_n \quad (25)$$

Letting the region I be the n th grid and region II, the m th grid, the left hand side of (23) becomes the left hand side of (12). Hence the reciprocal relation holds. Similarly we can write

$$\tau_{\xi\eta}^R = G(m, \xi\eta; n, \alpha\beta) e_{\alpha\beta}^P v_n + G(m, \xi\eta; n, \xi\eta) e_{\xi\eta}^P v_n \quad (26)$$

$G(m, \alpha\beta; n, \alpha\beta)$ and similar terms are the influence coefficients for residual stresses caused by inelastic strains in different grids. Eq. (16) gives the reciprocal relation of these influence coefficients. If the plastic strain is taken to be constant in each grid, Eq. (22) reduces to

$$G(x, \alpha\beta; n, \alpha\beta) = \frac{1}{v_n} \int_{v_n} G(x, \alpha\beta; \bar{x}, \alpha\beta) dv$$

and Eq. (24) reduces to

$$G(m, \alpha\beta; n, \alpha\beta) = \frac{1}{v_m} \int_{v_m} G(x, \alpha\beta; n, \alpha\beta) dv$$

If there are M grids with incremental $\Delta e_{\xi\eta}^P$ and N -grids with $\Delta e_{\alpha\beta}^P$. The residual stresses are written as

$$\Delta \tau_{\alpha\beta}^R = \sum_{n=1}^M G(m, \alpha\beta; n, \xi\eta) \Delta e_{\xi\eta}^P v_n + \sum_{n=1}^N G(m, \alpha\beta; n, \alpha\beta) \Delta e_{\alpha\beta}^P v_n \quad (27)$$

$$\Delta \tau_{\xi \eta m}^R = \sum_{n=1}^M G(m, \xi \eta; n, \xi \eta) \Delta e_{\xi \eta}^P v_n + \sum_{n=1}^N G(m, \xi \eta; n, \alpha \beta) \Delta e_{\alpha \beta}^P v_n \quad (28)$$

In finding the incremental plastic strains in different grids, solution of a system of simultaneous equations is required. Due to the reciprocal relation, the matrix of these equations become symmetrical. This helps greatly the numerical solution.

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